Large *Ku* regime problems

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1 Introduction

When we study the transport in turbulent plasmas, including the diffusion of stochastic magnetic-field lines and charged particles, it is inevitable to encounter the large *Kubo* number system ($Ku \approx \frac{l_{ac}}{\Delta_\perp}\frac{\delta B}{B_0}$, the ratio of field line's radial excursion from the unperturbed tori to its radial correlation length), in which the strong scattering control the time and space scales so that the linear theory is not available. In this

case, $\nabla \cdot \mathbf{B} = 0$ allows to write the magnetic field as a function ψ

$$\frac{dx}{dz} = b_r / B_0 = \frac{\partial \psi}{\partial y},$$
$$\frac{dy}{dz} = b_\theta / B_0 = -\frac{\partial \psi}{\partial x},$$

where field lines go along constant ψ. The turbulent diffusivity stemming from this expression is known as the magnetic-field-line diffusivity D_m. The behavior of the $\psi = const$ lines for a random ψ function then is equivalent to the diffusion of a passive tracer in a two-dimensional, steady, incompressible random flow which is governed by

$$\frac{d\vec{x}}{dt} = \nabla \psi \times \hat{z},$$

where ψ is a random stream function. Similarly, the streamlines of this flow are the contours of ψ. So the geometry of the contours of ψ should be the same for these two problems. If the Peclet number is large, $P \gg 1$, the transport shows long correlation phenomenon because the tracer particles advected along very large streamlines diffuse very slowly from these lines to more typical short closed lines and hence provide a significant coherent contribution to the turbulent diffusivity $D*$. This is consistent with the large Ku system. The larger the Peclet number, the longer and narrower the bundles of streamlines that dominate the effective transport in the considered flow (see Fig.1). The geometry of the streamlines is also relevant

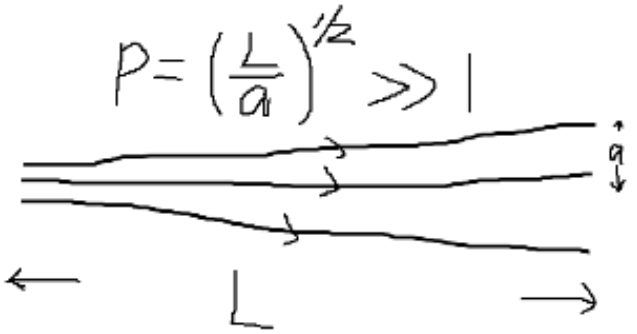

Figure 1: Cartoon for the streamlines in large Peclet number/Kubo number) system.

to the geometry of percolation clusters (Details could be found in **section 4**). So not only can our problem be associated with the transport in random media problem,

but also related to the percolation model. So in order to understand $Ku > 1$ regime, it is useful to examine the transport problem in random media and the percolation problem.

We consider three main problems here, the conductivity of a medium with spatially fluctuating properties (Dykhne problem), the advection diffusion of a tracer in an incompressible flow (Taylor problem and related shear dispersion), and basic percolation theory.

2 Conductivity of inhomogeneous media

For high *Ku* problem, effective transport coefficient in $2D$ random medium is of high interest. And the simplest example of such problem related to transport in random media is a random mixture of conducting. The conductivity of a two-phase thin film at equal concentrations of the phases and random distribution of them are studied by A. M. Dykhne in 1969 [1]. He considered a conducting medium consisting of parts of two types of arbitrary shape and dimensions, in which dimensions of the system are assumed to be much larger than the characteristic dimensions of the parts. If such a medium is placed in an electric field, currents flow through it and the pattern of the currents will be rather complicated, but it is believed that a qualitative analysis of scaling relations for the effective conductivity σ_{eff} is most important for the physical understanding of the problem.

Two-dimensional systems have a remarkable symmetry which allows one to calculate exactly the effective conductivity for arbitrary conductivities of the two phases. That it is possible to find an exact solution is due to the fact that the system of equations in the conditions described undergoes a symmetry transformation which does not change the macroscopic properties of the medium. So the method of *reciprocal media* is the key to this problem.

2.1 Conductivity of a two-phase system

The system of equations consists of Ohm's law $\mathbf{j} = \sigma\mathbf{e}$ and the equations of a constant current $\nabla \times \mathbf{e} = 0$, $\nabla \cdot \mathbf{j} = 0$. The conductivity σ is assumed to be given by a random function of the coordinates (x,y), taking two values; the regions (I,II) with the values $\sigma = \sigma_{1,2}$ are statistically equivalent. As we said, We are more interested in the effective conductivity, defining as

$$\mathbf{J} = \sigma_{eff}\mathbf{E}$$

where $\mathbf{J(E)} = V^{-1}\int \mathbf{j(e)}dV$ is the current(field) averaged over the system and σ_{eff} is the relevant effective conductivity. If the system is isotropic, we are allowed to

use an elegant reciprocity transformation

$$\begin{aligned} \mathbf{j}' &= (\sigma_1\sigma_2)^{1/2}\hat{n}\times\mathbf{e} \\ \mathbf{e}' &= (\sigma_1\sigma_2)^{-1/2}\hat{n}\times\mathbf{j} \end{aligned} \tag{1}$$

where $\hat{n}$ is the unit vector normal to the xy-plane. The transformation conserves the form of Ohm's law for the new current and field. And the averaged Ohm's law becomes

$$\mathbf{J}' = \sigma_{eff}\mathbf{E}'$$

with the same σ_{eff} based on the hypothesis that regions I and II are statistically equivalent. Then this gives us the effective conductivity as

$$\sigma_{eff} = (\sigma_1\sigma_2)^{1/2}. \tag{2}$$

Thus the logarithm of the conductivity is found to be additive on mixing or the effective is the geometric mean of σ_1, σ_2. This exact result is valid for arbitrary values of σ_1 and σ_2. Specifically, if σ_1 tends to infinity and σ_2 to zero, the effective conductivity may still remain finite. This is interesting result which is analogous to the percolation threshold. Furthermore, this result is consistent with the *quasilinear theory* in small fluctuation limit. Unfortunately, there is no universal formula for the effective conductivity of a material with an unequal amount of randomly distributed phases or with more than two phases, because the information about the volume fractions and the conductivities of the phases is not enough to determine the effective conductivity. The result will also depend on the spatial distribution of phases. Specifically, the conductivity behavior near the percolation threshold may depend crucially on the presence of long-range correlations in the distribution. It then becomes very complex.

2.2 General conductivity $\sigma(x,y)$

Now, we can generalize to more general conductivity which smoothly depends on the coordinates under such symmetry. For convenience we shall introduce the quantity $\chi(x,y) = \ln\sigma - \langle\ln\sigma\rangle$ and consider an ensemble of systems such that the conductivity distribution is an even function of the variables χ. Let us take a Gaussian distribution for the quantities χ. Again, substitution

$$\mathbf{j}' = \exp\{\langle\ln\sigma\rangle\}\,\hat{n}\times\mathbf{e},\ \mathbf{e}' = \exp\{-\langle\ln\sigma\rangle\}\,\hat{n}\times\mathbf{j} \tag{3}$$

won't change Ohm's law

$$\mathbf{j} = \exp(\langle\ln\sigma\rangle + \chi)\mathbf{e},\ \mathbf{j}' = \exp(\langle\ln\sigma\rangle - \chi)\mathbf{e}'$$

Replacing χ by $-\chi$ and using the fact that the distribution functions are even in χ, we again find that the primed system is macroscopically equivalent to the initial one.

Then we can repeat the argument of last section, yielding

$$\sigma_{eff} = \exp\langle \ln\sigma \rangle = \langle\sigma\rangle \exp(-\Delta^2/2)$$

for a Gaussian distribution, where $\Delta^2 = \langle \chi^2 \rangle$ is the root mean square fluctuation of the logarithm of the conductivity.

2.3 Current and field distribution characteristic

Certainly, this model makes it is possible to calculate other macroscopic characteristics of the distribution of currents and fields over the "phases and also within an individual phase. We shall calculate the average $\mathbf{A} = \langle(\sigma - \sigma_1)\mathbf{e}\rangle$. We can see that the expression being averaged is non-zero only in the second phase. Taking this into account, we find $\mathbf{A} = \frac{1}{2}(\sigma_2 - \sigma_1)\mathbf{E}_2$, where $\mathbf{E}_2 = V_2^{-1}\int \mathbf{e}_2 dV$ is the average field in the second phase. Expanding $\mathbf{A}$ term by term and using Eq.(2), we have

$$\mathbf{A} = (\sigma_{eff} - \sigma_1)\mathbf{E} = \frac{1}{2}(\sigma_2 - \sigma_1)\mathbf{E}_2.$$

This would give rise to the distribution of of fields as

$$\mathbf{E}_{1,2} = \frac{2\sqrt{\sigma_{2,1}}}{\sqrt{\sigma_1} + \sqrt{\sigma_2}}\mathbf{E} \tag{4}$$

The corresponding expression for the currents are also easily found

$$\mathbf{J}_{1,2} = \frac{2\sqrt{\sigma_{2,1}}}{\sqrt{\sigma_1} + \sqrt{\sigma_2}}\mathbf{J} \tag{5}$$

If we use $\mathbf{J}(\mathbf{J}_{1,2}) = \sigma_{eff}\mathbf{E}(\mathbf{E}_{1,2})$. To find the distribution over the phases of the energy being dissipated, let us use the expression

$$\sigma_1\sigma_2(\langle e^2\rangle_1 + \langle e^2\rangle_2) = \langle j^2\rangle_1 + \langle j^2\rangle_2 = \sigma_1^2\langle e^2\rangle_1 + \sigma_2^2\langle e^2\rangle_2.$$

Hence

$$\sigma_1\langle e^2\rangle_1 = \sigma_2\langle e^2\rangle_2 = (\sigma_1\sigma_2)^{1/2}\mathbf{E}^2 \tag{6}$$

The latter equality was obtained by using the relation

$$\frac{1}{2}(\sigma_1\langle e^2\rangle_1 + \sigma_2\langle e^2\rangle_2) = \langle(\mathbf{j}\cdot\mathbf{e})\rangle = (\mathbf{J}\cdot\mathbf{E})$$

Thus the energy is dissipated equally in the phases, regardless of the conductivities.

Using Eq.(6) we can calculate the mean square fluctuation characterizing the non-uniformity of the currents and fields in the system. We have

$$\begin{aligned}
\langle e^2\rangle &= \frac{1}{2}(\langle e^2\rangle_1 + \langle e^2\rangle_2) = \frac{1}{2}(\sqrt{\frac{\sigma_1}{\sigma_2}} + \sqrt{\frac{\sigma_2}{\sigma_1}})\mathbf{E}^2, \\
\langle j^2\rangle &= \frac{1}{2}(\sqrt{\frac{\sigma_1}{\sigma_2}} + \sqrt{\frac{\sigma_2}{\sigma_1}})\mathbf{J}^2, \\
\frac{\langle j^2\rangle - \mathbf{J}^2}{\mathbf{J}^2} &= \frac{\langle e^2\rangle - \mathbf{E}^2}{\mathbf{E}^2} = \frac{1}{2}[(\frac{\sigma_1}{\sigma_2})^{1/4} - (\frac{\sigma_2}{\sigma_1})^{1/4}]^2.
\end{aligned} \tag{7}$$

3 Advective-diffusion transport

For a pure $2D$ problem, $\nabla \cdot \mathbf{B}$ ensures that all the field lines in domain are closed. So it is of interest to consider the scattering from closed loop to closed loop for large Ku system. This leads us to study the cell-layer problem. So in this section we will focus on the effective diffusion of a passive component (such as test particles, dye, temperature, etc.) in a laminar flow pattern when local diffusion is present, with a diffusivity D attributable to collisions or fluctuations. The basic physical picture here is medium of neighboring non-overlapping convective cells.

In this model, each passively convected component has its own characteristic diffusivity D in the rest frame of the material. And the actual transport of the passive component driven by a large-scale gradient gives rise to an enhanced effective diffusivity $D*$ as a result of the increased transport from the convective motion. There exist several classes of flows, whose transport properties can be calculated exactly. The simplest example of a shear flow in a tube was considered by Taylor (1953,1954) [2], who found the effective coefficient of the diffusivity for the Poiseuille flow is given by $D* = a^2 v_0^2/(192D)$ for a large Peclet number ($P = v_0 a/D$) system, where v_0 is the flow velocity of convective cells with characteristic size a. Aris (1956) [3] gave a more general solution of the enhancement of diffusivity to the case of arbitrary Peclet number and different tube cross sections, where all possibilities are covered by one exact formula

$$D* = D + K\frac{a^2 v_0^2}{D} \tag{8}$$

with different values of the numerical constant K.

3.1 Estimate of the effective diffusivity

The behavior of effective diffusivity in the large-Peclet-number limit depends on the topology of flow streamlines. The best mixing properties are exhibited by flows with extended streamlines, such as shear flows, where the mixing length tends to infinity as $D \to 0$. For example, we can estimate the effective diffusion in a simple laminar periodic flow pattern [4]. The governing equation for the density of the dye in the incompressible flow is

$$\frac{\partial n}{\partial t} + \mathbf{u} \cdot \nabla n = D\nabla^2 n, \tag{9}$$

where n is the density of a passively advected agent (concentration of an impurity, temperature, etc.), D is the molecular diffusion coefficient and u the given fluid velocity which for a single mode is of the form

$$\mathbf{u} = (\tilde{u}d/\pi)\hat{z} \times \nabla\psi$$

with the streamline function ψ. Thus d gives the size of a roll and $\tilde{u}$ the maximum flow velocity (Here we are interested in the effective transport coefficient for scales $L \gg d$). Fig.2 shows a segment of the flow pattern.

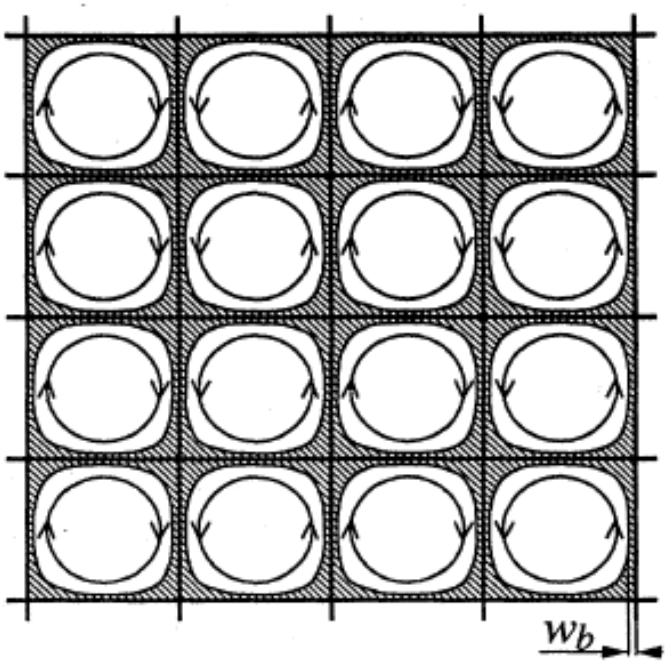

Figure 2: Periodic system of convection rolls. The hatched region denotes diffusive boundary layers near the separatrices, where the tracer density gradients are localized in the high-Peclet number limit.

The micro-structure introduced by the flow is characterized by two characteristic times: $\tau_H = d/\tilde{u}$ the time for circulation around the roll, and $\tau_D = d^2/D$ the time for molecular diffusion of a particle trough a roll. The ratio τ_D/τ_H is the known Peclet number P. Here we consider the interesting case $P \gg 1$. Then we can expect that the effective diffusion is a hybrid of two precess, the convection operated in cells and the diffusion operated in boundary layers.

Consider the particles within a given roll, and how they will move during a time τ_H. Those confined in the interior of a roll will simply circulate around it. However, those close to the edge in a boundary layer may diffuse across a roll boundary, after which they are effectively convected a distance d in a random direction depending on which boundary is crossed. Thus, the effective diffusion coefficient for the random walk is

$$D* = fd^2\tau_H^{-1},$$

where f is the fraction of particles in the boundary layer. To estimate f we note that it is specified by the number of particles close enough to the roll boundary to diffuse

to the neighbouring roll in a circulation time. Thus, $f = \delta/d$ with the boundary layer thickness specified by $\delta^2 = D\tau_H$. This leads to an estimate for the effective diffusion coefficient

$$D* \approx (D\bar{u}d)^{1/2} = DP^{1/2} \tag{10}$$

3.2 Physical picture

As we mentioned above, the effective diffusion is hybrid of a fast kicks through cells and a slow diffusion through boundary layers. Precisely speaking, the effective diffusivity is geometric mean of slow D and fast $D_{cell}(ud)$. To indicate again the physical nature of the solutions, we show in Fig.3 a schematic plot of the density versus x for fixed y. The solid curve would apply for all y except the horizontal boundaries, the dotted curve would apply at the boundaries.

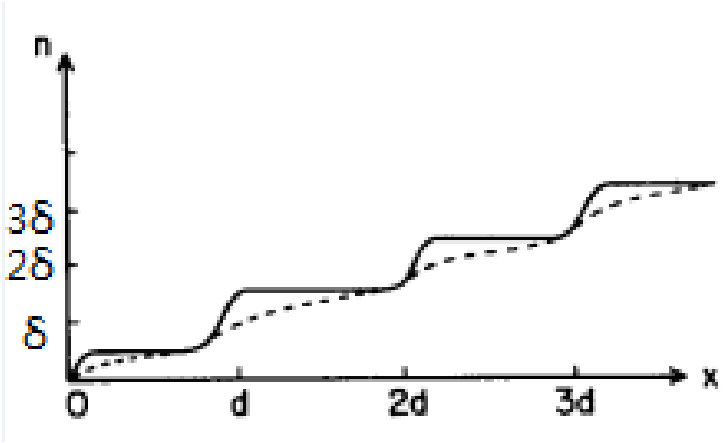

Figure 3: Schematic density profile for the dye along the x direction. Steep transitions in the density exist between each cell due to boundary layer diffusion.

The main point, of course, is that the density must be flat in the interior of the rolls because of the rapid circulation so that the global gradient must appear as a steep gradient confined to the boundary layer. This steep gradient then leads to an enhanced diffusive flux as given by Eq.(10), i.e, the transport appears only at the boundary layer and the molecular diffusion the the source of irreversibility of this system.

3.3 Shear dispersion

In Taylor's work, he gave a simple recipe for the calculation of contaminant dispersion in bounded shear flows at large times after discharge, where he revealed the mechanism whereby velocity shear begins to pull out a heat or dye spot and leads to a rapidly increasing rate of dilution. He recognized that for bounded shear flows

this shear dispersion mechanism continues to operate, even when the concentration has become nearly uniform across the flow. As long as there is some concentration variation across the flow, the different velocities in different parts of the flow $u = u_0(1 - r^2/a^2)$ proved an efficient mechanism for longitudinal dispersion (see Fig.4). The differential advection stretches the tracer cloud, causing it to spread longitudinally more quickly than a cloud released into a uniform current. In addition, differential advection creates lateral gradients in concentration, which, when acted upon by lateral diffusion, accelerates the dilution of the cloud. So the effective diffusivity must be enhanced by the flow velocity.

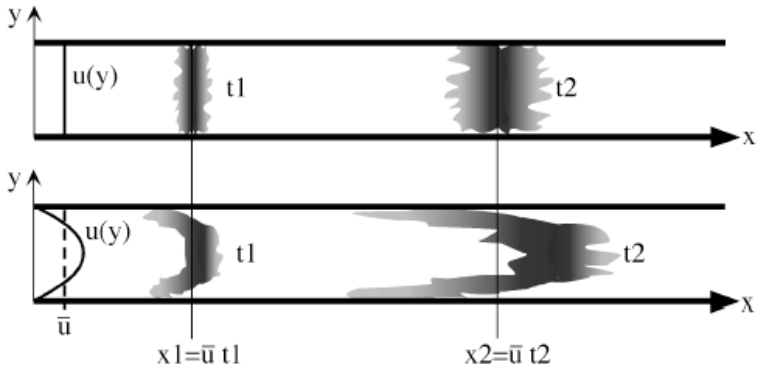

Figure 4: Top-view of two channels. In the top channel the side-wall boundary condition allows slip, and the velocity profile is uniform across y. In the bottom channel there is a no-slip condition at the side-wall boundary, creating lateral shear, $\partial u/\partial y$. Both channels have the same cross-sectional mean velocity, u. Tracer is released at t = 0, x = 0. The spatial distribution of each cloud is shown above at time t1 and t2. Velocity shear in the bottom channel stretches the tracer cloud, and as a result this cloud spreads longitudinally more rapidly than the cloud in the uniform channel.

For this case, we expect that $D_{eff} = D_0 + D_{shear}$. And we need to estimate D_{shear} here. The distortion observed in the bottom channel of Fig.3 can be predicted from the full transport equation of the concentration, saying c,

$$\frac{\partial c}{\partial t} + \mathbf{v} \cdot \nabla c = D_0 \nabla^2 c \tag{11}$$

The key to Taylor's analysis was the calculation of the residual concentration variation across the flow. So here we use a method much like quasilinear theory,

defining

$$c = \langle c \rangle + \tilde{c},\ \langle c \rangle = \frac{1}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}} c(x,y)dy$$
$$\mathbf{v} = \langle \mathbf{v} \rangle + \tilde{\mathbf{v}},\ \langle \mathbf{v} \rangle = \frac{1}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}} \mathbf{v}(x,y)dy.$$

We can see that deviations from the depth-average are denoted by a tilde. Plugging them into Eq.(11) yields

$$\frac{\partial \langle c \rangle}{\partial t} + \frac{\partial \tilde{c}}{\partial t} + \langle \mathbf{v} \rangle \cdot \nabla \langle c \rangle + \tilde{\mathbf{v}} \cdot \nabla \langle c \rangle + \langle \mathbf{v} \rangle \cdot \nabla \tilde{c} + \tilde{\mathbf{v}} \cdot \nabla \tilde{c} = D_0 \nabla^2 (\langle c \rangle + \tilde{c})$$

Analogues to the quasilinear theory, we take the We take the depth-average of each term, yielding

$$\frac{\partial \langle c \rangle}{\partial t} + \langle \mathbf{v} \rangle \cdot \nabla \langle c \rangle + \langle \tilde{\mathbf{v}} \cdot \nabla \tilde{c} \rangle = D_0 \nabla^2 \langle c \rangle$$

We can see that a new term has appeared, $\langle \tilde{\mathbf{v}} \cdot \nabla \tilde{c} \rangle$, that represents the flux associated with the correlation between the fluctuations in velocity,$\tilde{u}$, and concentration $\tilde{c}$, relative to their depth-averaged values. With the next few steps we will find a solution for $\tilde{c}$, so that we might evaluate this new flux term. We can get equation for $\tilde{c}$ as

$$\frac{\partial \tilde{c}}{\partial t} + \tilde{\mathbf{v}} \cdot \nabla \langle c \rangle + \tilde{\mathbf{v}} \cdot \nabla \tilde{c} - \langle \tilde{\mathbf{v}} \cdot \nabla \tilde{c} \rangle = D_0 \nabla^2 \tilde{c}$$

where we have changed to the moving frame with $\langle \mathbf{v} \rangle$. Now we place an important condition on the magnitude of the spatial fluctuation $\tilde{c}$. Specifically, we assume that $\tilde{c} \ll \langle c \rangle$. If this is true, then the magnitude of both the third and fourth terms in Eq.(12) must be smaller compared to the second term, and we may drop the small terms, reducing it to

$$\frac{\partial \tilde{c}}{\partial t} + \tilde{\mathbf{v}} \cdot \nabla \langle c \rangle = D_0 \nabla^2 \tilde{c}. \tag{12}$$

This equation describes how vertical fluctuations in concentration are created by differential advection and destroyed by vertical diffusion. Let seek the stationary solution, i.e., differential advection is balanced by the vertical diffusion. Then it becomes

$$\tilde{\mathbf{v}} \cdot \nabla \langle c \rangle = D_0 \nabla^2 \tilde{c}.$$

Take $\nabla^2 \tilde{c} = -k_y^2 \tilde{c}_{ky}$ that $k_y^2 > (2\pi)^2 / a^2$. Then we can get the expression for the concentration perturbation

$$\tilde{c}_{ky} = -\frac{\tilde{v_x}}{k_y^2 D_0} \frac{\partial \langle c \rangle}{\partial x}.$$

Then the new term due to the mixing of molecular diffusion and shear of the flow becomes

$$\langle \tilde{v_x}\tilde{c}\rangle = -\sum_{k_y} \frac{\left|\tilde{v_{k_y}}\right|^2}{k_y^2 D_0}\frac{\partial\langle c\rangle}{\partial x} = -D_{flow}\frac{\partial\langle c\rangle}{\partial x}.$$

So the effective axial diffusivity is given by

$$D = D_0 + \sum_{k_y} \frac{\left|\tilde{v_{k_y}}\right|^2}{k_y^2 D_0}. \tag{13}$$

We can see that this result is similar to the problem in former section that the effective diffusion for axial spreading is the combination of the laminar flow and molecular diffusion. This is more clear when $\tilde{v}$ is comparable with mean flow velocity, which gives us

$$D = D_0 + \alpha\frac{v_0^2 a^2}{D_0}. \tag{14}$$

As we can see, this result is akin to the cell problem, i.e., the laminar flow combining with the molecular diffusion yields the transport.

4 Percolation problem

The percolation problem describes the simplest possible phase transition with nontrivial critical behavior [5]. The general formulation of the percolation problem is concerned with elementary geometrical objects (spheres, sticks, sites, bonds, etc.) placed at random in a d-dimensional lattice or continuum. One is interested in how many objects can form a cluster of communication and, especially, when and how the clusters become infinite. Correspondingly, the *percolation threshold*, is the minimum concentration at which an infinite cluster spans the space. And this model is relevant for a number of transport problems. In the regime we are interested, the geometry of the streamlines ψ is associated with the geometry of percolation clusters as follows. Let us call "objects" the regions where $\psi(x,y)$ is less than a specified constant level h. If $z = \psi(x,y)$ is imagined to be the elevation of a random landscape and h designates the level of flooding, then the objects are the lakes (see Fig.5). Two neighboring lakes "communicate" if they merge into a bigger lake, which is a "cluster". So the contours $\psi(x,y) = h$ present the coastlines of the lakes, that is, the envelopes of the clusters. Then the random contours problem is related to the formation of clusters of the random objects in percolation theory. The control parameter of this percolation problem is the level h such that at some critical level, $h = h_c$ the lakes form an infinite ocean and among the contours $\psi(x,y) = h_c$, there is at least one infinitely long. But the effective diffusion in a random flow presents an example of a long-range correlated phenomenon ($P \gg 1$) without critical behavior.

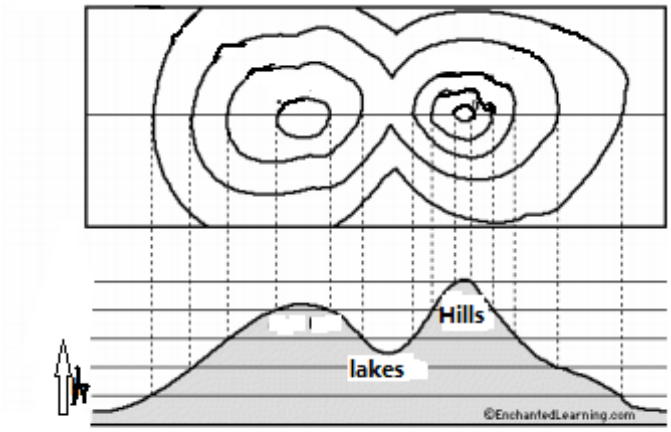

Figure 5: Topographical map of streamlines ψ. Lines traverse along constant ψ contours. Most contours are closed, isolated, thus they make little contribution to the transport. But the contours along "passes" can take a long path lengths on along which transport occurs primarily.

The critical exponents of the percolation transition enter the result because the large value of the control parameter ($P \gg 1$)) picks up a near-critical (in the sense of the contour percolation) set of streamlines dominating the effective transport.

In two dimensions the basic percolation exponents are known exactly thereby presenting all necessary characteristics of the long-range contour behavior. The pure geometrical nature of this transition makes it useful for us to study large Ku system.

4.1 What is the difference of percolation and diffusion?

There are many physical phenomena in which a fluid spreads randomly through a medium. Besides the random mechanism, external forces may govern the process, as with water percolating through limestone under gravity. According to the nature of the problem, it may be natural to ascribe the random mechanism either to the fluid or to the medium. Most mathematical analyses are confined to the former alternative, for which we retain the usual name of diffusion process: in contrast, there is some work on the latter alternative, which we shall call a percolation process [6]. Here we would list some examples to illustrate the differences between diffusion and percolation.

4.1.1 Example 1

The simplest example of a diffusion process is the one-dimensional Polya walk. In this, a particle (the fluid) takes steps of unit length along a straight line (the medium) starting from the origin. After any number of steps, the particle has, independently of its previous history, equal probabilities (each $\frac{1}{2}$) of taking its next step to the right or to the left. As is well known, the position of the particle after n steps is then a linear transformation of a binomial variate and has a distribution with zero mean and variance n. When n is infinite, the particle visits every point of the medium infinitely often with probability 1.

In the percolation process which is analogous to diffusion process above, fluid and medium are the same as before; but the stochastic mechanism resides in the medium rather than in the particle. Specifically, each point of the medium has, independently of the other points, equal probabilities (each $\frac{1}{2}$) of being a 'right-sense' or a 'left-sense' point. The particle starts from the origin and takes steps of unit length, the direction of any step being that of the sense of the point from which that step starts. Thus the state of the medium entirely determines the motion of the particle, which moves steadily in one direction until it encounters successive points of opposite sense, whereupon it oscillates between them. The distribution of terminal position is nothing like binomial, and it has zero mean and variance $\frac{1}{18}(81-(-1)^n-(3n+5)(\frac{1}{2})^{n-4})$ after n steps. When n is infinite, there is probability 1 that the particle will visit only finitely many points.

Briefly speaking, the medium is deterministic while motion is stochastic in diffusion. But percolation process has deterministic motion and stochastic medium.

4.1.2 Example 2

Some physical situations may be regarded either as diffusion or as percolation processes. Suppose each individual in a branching (or cascade) process has, independently of the other individuals, respective probabilities $q^2, 2pq, p^2$ of giving birth to 0, 1, 2 descendants in the next generation. We may visualize this as a diffusion process by thinking of a branching fluid advancing from generation to generation: any one branch of the fluid at any generation carries with it a random mechanism that decides whether it provides 0, 1 or 2 branches of fluid in the next generation. But equally, we may think of a system of channels leading from the original ancestor such that each channel divides into precisely two channels at each generation. Each of these channels has, independently of the other channels, a probability q of being dammed. This random set of dams in the channels (the medium) will determine how fluid introduced at the ancestor will spread; this description is now a percolation process. It is a well known result that, if $p \text{`} \frac{1}{2}$, only finitely many channels will be wet by the fluid with probability 1, and that the corresponding probability of ultimate extinction is q^2/p^2 when p exceeds the critical value $\frac{1}{2}$. Critical probabilities play

similar roles in more general percolation processes, as we shall see later.

4.2 General aspects

Generally, the intrinsic and the random characteristics of the medium, together with any external laws which may operate, completely determine the progress of the fluid. The intrinsic characteristics of the medium consist in its interconnecting structure. The random characteristics of the medium are introduced by randomly damming some of its connexions. The resultant system will be called a random maze. Fluid supplied at various points flows along all the undammed paths. That is The fluid will be able to flow from one point to another if and only if there is a connection without dams between them, and this will be so if and only if there is an undammed self-avoiding walk connecting them.

We deal with abstract objects called sites and bonds. A bond is a path between two sites, and may either be two-way or may permit a walk in one direction only (see Fig.6). Suppose that in an infinite set of sites joined by bonds some (or all)

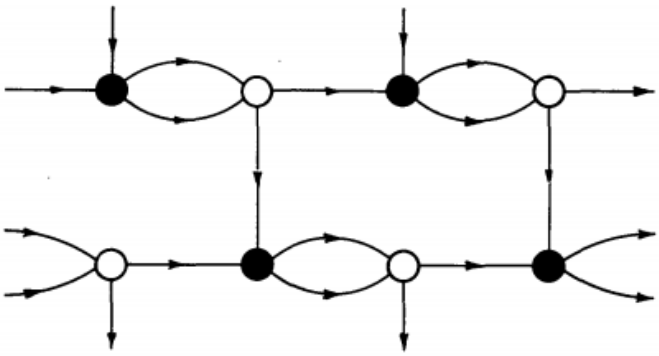

Figure 6: An example of random maze

of the bonds are dammed in a random manner. An site of the set is said to be wet by the fluid either if it is a source site or if there exists a walk to the site from a source site, the walk traversing undammed bonds only and in the permitted directions. All sites not wet are said to be dry. Given a periodic lattice embedded in a d-dimensional space and the probability p for each site of the lattice to be wet, percolation theory is specially concerned with the percolation threshold $p = p_c$, at which an *infinite cluster* spans the lattice. Here a cluster means a conglomerate of wet s sites, which communicate via the nearest-neighbor rule (see Fig.7). Besides this *site percolation*, one can introduce the bond percolation, with clusters of connected conducting bonds (see Fig.8). The site and the bond percolation problems are very similar to each other. To be specific, we discuss lattice lattice percolation mainly with the example of the site problem. Consider a lattice of N sites where $N \gg 1$ and define the concentration of wet sites as x (note that x has the same

Figure 7: 2D triangular lattice bond percolation clusters at critical $p_c = 0.3473$

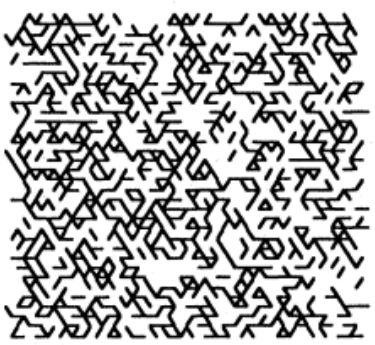

Figure 8: 2D square lattice site percolation clusters at critical $p_c = 0.5927$

meaning of p mentioned above). Then we have the following results for $N \gg 1$:

1. **If $x \ll x_c$, only small isolated clusters exist,**
2. **If x increases, larger clusters would form,**
3. **There exists a critical x_c at which an infinite cluster definitely appears.**
4. **Critical probability decreases with increasing dimension d**
5. **There exists exactly one ($x \geq x_c$) or no ($x < x_c$) infinite cluster for 2D/3D.**

This infinite cluster is characterized by the density $P^s(x)$, which denotes the probability for a given site to belong to the infinite cluster ($P^s(x) = \frac{N_{cluster}}{N}$). In the vicinity of the percolation threshold, the function $P^s(x)$ is non-analytic. There is extensive numerical evidence for the power dependence

$$P^s(x) \propto (x - x_c)^s \theta(x - x_c), \quad |x - x_c| \ll 1, \tag{15}$$

where $\theta(x)$ is the Heaviside step function. The exponent β is one of the standard set of *critical exponents* that govern the behavior of different quantities near the critical point. Its value has been studied in many works. It is summarized in [5]. These exponents depend only on the dimension of the space and not on the type of lattice or the kind of percolation problem.

5 Conclusion

The idea of magnetic confinement fusion is to organize the geometry of the magnetic field (e.g., nested toroidal magnetic surfaces) in such a way that $\chi_\perp$ controls the heat losses from plasma. In reality, however, there may arise irregularities of **B**, which, due to the triggering of the large longitudinal conductivity $\chi_\parallel$, can significantly increase the effective heat conductivity $\chi_\perp^*$ across the unperturbed magnetic field [7].

In a two-dimensional magnetic field, $\mathbf{B} = \nabla\psi(x,y) \times \hat{z}$, the reciprocity theorem is still valid (see section 2). Hence in the macroscopically isotropic case the effective heat conductivity is given by [8]

$$\chi^* = (\chi_\parallel \chi_\perp)^{1/2}. \tag{16}$$

Ref. [5] also talked about the "Magnetoresistance of inhomogeneous media with the Hall effect", where he used the method mentioned in section 2 and 3. In the case of stronger magnetic perturbations, the magnetic line diffusivity is given by the percolation scaling

$$D_m \approx \lambda_\perp \frac{\delta B}{B_0} R^{-1/(\nu d_h + 1)} \propto [\frac{\delta B}{B_0}]^{0.7}$$

There are more applications of these theorems to the problems of transport in turbulent plasmas that we can learn in the future.

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